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Some calculations on a time-of-flight β-spectrometer with trochoidal orbits

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With 1 figure in the text

Introduction

Magnetic β-spectrometers are characterized by two essential properties, the resolution and the transmission. In all such spectrometers improved resolution can be obtained at the expense of the transmission and vice versa. As a rule 1 % resolution cannot be combined with more than one or possibly a few per cent transmisson. A spectrum is measured point for point by varying the magnetic field, except in the case of photographic recording as in the semicircular spectrometer.

In *scintillation spectrometers* conditions are somewhat different. The detection probability, corresponding to the transmission, can be very high, but on the other hand, the energy resolution by means of pulse height analysis cannot be made better than 5–10 %, a limit which is mainly set by the present state of the development of photomultipliers. By using a multichannel analyzer, a whole scintillation spectrum, or a certain part of it, can be measured simultaneously, which is, of course, a great advantage especially in the case of a rapidly decaying source or in the study of radiation induced by an accelerator.

It seems possible to combine some of the features of these complementary types of spectrometer by introducing the time-of-flight technique into β-spectroscopy. The velocity of a β-particle is close to the velocity of light, which means that a direct determination of the particle velocity will give very poor information about the energy. By introducing the trochoidal motion in an inhomogeneous magnetic field, this difficulty is overcome because in this case the flight path is multiplied by a large factor which depends on the particle energy. The principle was mentioned briefly in an earlier report [1] and it is the purpose of this paper to discuss in more detail the expected properties of such a spectrometer.

I. On the motion of a charged particle in a magnetic field with rotational symmetry and the radial dependence $1/r^n$

It is the purpose of this chapter to study the motion of a charged particle in a magnetic field with rotational symmetry and the radial dependence $1/r^n$.

In the first part the problem will be confined to such particles which move in the plane of symmetry of the magnet. In this case the motion can be calculated in a conventional way.

The actual motion is, however, a superposition of a trochoidal motion and an oscillation through the symmetry plane. This problem will be treated by means of a perturbation method developed by Alfvén.

A: Let us consider a particle which starts radially at $r = r_0$, and let ϱ_0 be the radius of curvature at this point. The motion is then determined by the following differential equation:

$$\frac{\left[r^2 + \left(\frac{dr}{d\varphi}\right)^2\right]^{3/2}}{r^2 + 2\left(\frac{dr}{d\varphi}\right)^2 - r \cdot \frac{d^2 r}{d\varphi^2}} = \varrho_0 \cdot \left(\frac{r}{r_0}\right)^n \tag{1}$$

By means of the transformation:

$$\frac{dr}{d\varphi} = r \cdot \tan\alpha \tag{2}$$

this can be integrated and gives:

$$\varepsilon \cos\alpha = \frac{1}{n-2} \cdot \left\{\frac{r_0}{r} - \left(\frac{r_0}{r}\right)^{n-1}\right\} \quad \text{where} \quad \varepsilon = \frac{\varrho_0}{r_0} \tag{3}$$

A combination of (2) and (3) gives:

$$d\varphi = \frac{\varepsilon \cos\alpha \, d\alpha}{(n-1) \cdot \varepsilon \cos\alpha - \frac{r_0}{r}} \tag{4}$$

From (3) we can find $\frac{r_0}{r}$ as a series expansion in powers of $\varepsilon \cos\alpha$

$$\frac{r_0}{r} = 1 - \varepsilon \cos\alpha - \frac{n-1}{2} \cdot (\varepsilon \cos\alpha)^2 - \frac{n(n-1)}{3} \cdot (\varepsilon \cos\alpha)^3 + \cdots$$

which makes possible the integration of (4)

$$\Phi = \pi \cdot n \cdot \frac{\varrho_0^2}{r_0^2} \cdot \left\{1 + \left(\frac{3}{4} n^2 + n - 1\right) \varepsilon^2 + \cdots\right\} \tag{5}$$

Φ is the angular displacement of the starting point after one revolution.

In an analogous manner we can calculate the length, L, of the particle orbit during one revolution:

$$dl = \frac{\varepsilon r_0 \, d\alpha}{\frac{r_0}{r}\left\{(n-1) \cdot \varepsilon \cos\alpha - \frac{r_0}{r}\right\}} \tag{6}$$

$$L = 2\pi \varrho_0 \cdot \left\{1 + \frac{n^2 + 2n}{2} \cdot \varepsilon^2 + \cdots\right\} \tag{7}$$

If the particle velocity (which is unchanged during the motion) is called v we find for the drift velocity:

$$v_{\text{drift}} = \frac{r_0 \Phi}{L/v} = \frac{n}{2} \cdot \frac{\varrho_0}{r_0} \cdot v \cdot \left(1 + \frac{n^2-4}{4} \cdot \varepsilon^2 + \cdots\right) \tag{8}$$

Here we can insert the relativistic expressions for ϱ_0 and v and find:

$$v_{\text{drift}} = c \cdot \frac{n}{r_0} \cdot \frac{mc}{eH_0} \cdot E \cdot \frac{1+E/2}{1+E} \cdot \left(1 + \frac{n^2-4}{4} \cdot \frac{\varrho_0^2}{r_0^2} + \cdots\right) \tag{9}$$

where E is the particle energy in units of mc^2.

It is interesting to observe that the correction term, which is small if $\varrho_0 \ll r_0$ vanishes if $n=2$.

B: A perturbation method for studying the motion of a charged particle in a magnetic field has been developed by Alfvén [2]. The method is applicable in cases where the particle momentum is sufficiently low, $H\varrho \ll H^2/\text{grad } H$. (In the actual case this condition can be written $\varrho \ll r/n$.) The particle then moves essentially in a circle which is subjected to a drift motion due to the inhomogeneity of the field, at the same time as there is a motion along the magnetic lines of force. The circular motion is equivalent to a small magnetic dipole with the moment μ.

$$\mu = \frac{\mathcal{E}_\perp}{H}, \quad \text{where} \quad \mathcal{E}_\perp = \frac{m v_\perp^2}{2}$$

and $v_\perp$ is that component of the particle velocity which is perpendicular to the magnetic field. It can be shown that μ does not change during the motion. Alfvén has applied the method to a study of the motion of a charged particle in a magnetic dipole field, in which case the field decreases in the symmetry plane as $1/r^3$.

It is the purpose here to extend such calculations to the general case of a magnetic field with rotational symmetry and the radial dependence $1/r^n$.

In a cylindrical coordinate system the first terms in a series expansion of the field components H_z and H_r are the following, determined by the conditions rot $H=0$ and div $H=0$

$$H_z = \frac{a}{r^n} \cdot \left(1 - \frac{n^2}{2} \cdot \frac{z^2}{r^2} + \frac{n^2(n+2)^2}{24} \cdot \frac{z^4}{r^4} + \cdots\right) \tag{10}$$

$$H_r = -\frac{a}{r^n} \cdot \frac{nz}{r} \cdot \left(1 - \frac{n(n+2)}{6} \cdot \frac{z^2}{r^2} + \frac{n(n+2)^2(n+4)}{120} \cdot \frac{z^4}{r^4} + \cdots\right) \tag{11}$$

It is now possible to calculate

$$H, \frac{\partial H}{\partial z}, \frac{\partial H}{\partial r}, \frac{H_z}{H}, \frac{H_r}{H} \quad \text{and} \quad \frac{H_z}{H_r},$$

expressions which will be needed in the following. Further it is necessary to have the equation of a magnetic line of force intersecting the plane of symmetry at radius r_0. The line of force is defined by the differential equation $\frac{dz}{dr}=\frac{H_z}{H_r}$ which gives:

$$r=r_0\cdot\left(1-\frac{n}{2}\cdot\frac{z^2}{r_0^2}-\frac{n^2(2n+1)}{24}\cdot\frac{z^4}{r_0^4}+\cdots\right) \tag{12}$$

The force acting upon the equivalent dipole is a magnetic force with the components

$$f_r^{(m)}=-\mu\cdot\frac{\partial H}{\partial r} \quad \text{and} \quad f_z^{(m)}=-\mu\cdot\frac{\partial H}{\partial z},$$

which gives for the component perpendicular to the magnetic field:

$$-\frac{1}{\mu}\cdot f_\perp^{(m)}=\frac{H_r}{H}\cdot\frac{\partial H}{\partial z}-\frac{H_z}{H}\cdot\frac{\partial H}{\partial r} \tag{13}$$

In addition to magnetic forces there is also the centrifugal force $f^{(c)}$ which is parallel to $f_\perp^{(m)}$.

$f^{(c)}=\frac{2\mathcal{E}_{\parallel}}{R}$, where R is the radius of curvature of the magnetic line of force. Since the component of grad H along the radius of curvature is equal to H/R, and since this is also equal to $\frac{1}{\mu}f_\perp^{(m)}$, we find the following expression for the centrifugal force:

$$f^{(c)}=\frac{2\mathcal{E}_{\parallel}}{\mathcal{E}_\perp}\cdot f_\perp^{(m)} \tag{14}$$

The motion of the equivalent dipole is determined by the following two equations:

$$\frac{dz}{dt}=\frac{H_z}{H}\cdot v_{\parallel}=\frac{H_z}{H}\cdot\sqrt{\frac{2\mu}{m}(H_1-H)} \tag{15}$$

$$r\cdot\frac{d\lambda}{dt}=-\frac{1}{eH}(\mathrm{f}_\perp^{(m)}+f^{(c)})=\frac{\mu}{eH}\cdot\frac{2H_1-H}{H}\cdot\left(\frac{H_r}{H}\cdot\frac{\partial H}{\partial z}-\frac{H_z}{H}\cdot\frac{\partial H}{\partial r}\right) \tag{16}$$

where λ is the aximuthal angle.

The first equation which describes the component of the motion along a line of force is directly integrable since H and H_z can be expressed explicitly as functions of z and r_0. H_1 is a constant which corresponds to the maximum field strenght which is attained at the maximum oscillation amplitude $z=\pm z_1$.

In (15) and (16) the value of μ, which is a constant, has to be inserted.

$$\mu=\frac{1}{2}\cdot e\cdot v\cdot\varrho_1, \tag{17}$$

where v is the particle velocity and ϱ_1 is the mean radius of curvature at the turning point ($z=z_1$), where v is perpendicular to the magnetic field. This can also be written

$$\mu=\frac{1}{2}\cdot e\cdot v\cdot \varrho_0\cdot\frac{H_0}{H_1}=\frac{1}{2}\cdot e\cdot v\cdot\varrho_0\cdot\left(1-\frac{n^2}{2}\cdot\frac{z_1^2}{r_0^2}+\cdots\right) \tag{18}$$

where ϱ_0 is the mean radius of curvature in the plane of symmetry of the magnet if the oscillation amplitude were zero.

The integration of (15) gives:

$$T=\frac{2\pi r_0}{n}\cdot\frac{\sqrt{2}}{v}\cdot\left(1+\frac{3n^2-4}{16}\cdot\frac{z_1^2}{r_0^2}+\cdots\right) \tag{19}$$

where T is the time of one complete oscillation. A combination of (15) and (16) gives after integration:

$$\Lambda=\pi\cdot\sqrt{2}\cdot\frac{\varrho_0}{r_0}\cdot\left(1-\frac{n^2-8n+4}{16}\cdot\frac{z_1^2}{r_0^2}+\cdots\right) \tag{20}$$

where Λ is the azimuthal angle corresponding to one complete oscillation. The drift velocity can now be obtained from the combination of (19) and (20).

$$v_{\text{drift}}=\frac{r_0\cdot\Lambda}{T}=\frac{n}{2}\cdot\frac{\varrho_0}{r_0}\cdot v\cdot\left(1+\frac{n(2-n)}{4}\cdot\frac{z_1^2}{r_0^2}+\cdots\right) \tag{21}$$

in conformity with the result of the preceding calculation, eq. (8).

The method used here is applicable in the case of non-relativistic particles, but since the true trochoidal orbit is determined only by magnetic forces, eq. (20), which is a geometrical relation, must be valid also in the relativistic case. The interpretation of eq. (21) is that the flight path in the trochoidal orbit is $\frac{2r_0}{n\varrho_0}$ times longer than the drift movement. This means that eq. (21) and thus eq. (19) are valid also in the relativistic region.

If the particle energy is expressed in units of mc^2, the equations can be written in the following form:

$$T=\frac{2\pi r_0}{n\cdot c}\cdot\frac{1+E}{\sqrt{E}\cdot\sqrt{1+E/2}}\cdot\left(1+\frac{3n^2-4}{16}\cdot\frac{z_1^2}{r_0^2}+\cdots\right) \tag{19'}$$

$$\Lambda=\frac{2\pi}{r_0}\cdot\frac{mc}{eH_0}\cdot\sqrt{E}\cdot\sqrt{1+E/2}\cdot\left(1-\frac{n^2-8n+4}{16}\cdot\frac{z_1^2}{r_0^2}+\cdots\right) \tag{20'}$$

$$v_{\text{drift}}=c\cdot\frac{n}{r_0}\cdot\frac{mc}{eH_0}\cdot E\cdot\frac{1+E/2}{1+E}\cdot\left(1+\frac{n(2-n)}{4}\cdot\frac{z_1^2}{r_0^2}+\cdots\right) \tag{21'}$$

II. On the possibility of applying the trochoidal particle motion to β-spectroscopy

As seen from (21′) the drift velocity is essentially proportional to the particle energy, a fact which opens the possibility of measuring β-energies by the time-of-flight technique. The time taken for a particle to proceed one half turn round the magnet is:

$$\tau = \frac{\pi r_0}{v_{\text{drift}}} = \frac{\pi \cdot e}{m c^2} \cdot \frac{H_0 \cdot r_0^2}{n} \cdot \frac{E+1}{E(1+E/2)} \cdot \left(1 + \frac{n(n-2)}{4} \cdot \frac{z_1^2}{r_0^2} + \cdots\right) \tag{22}$$

If $n=2$ the following important properties are achieved

1) The flight time is independent of the mean radius r_0, since $H_0 \sim \frac{1}{r_0^2}$.

2) The flight time is, to the first approximation, independent of the oscillation amplitude, because the correction term vanishes.

3) The energy correction term due to the finite size of ϱ_0/r_0 vanishes in eq. (9).

4) With a reasonable magnet ($r_0 = 50$ cm, $H_0 = 12\,000$ oersteds) and at medium particle energy (~ 1 MeV), the flight time is of the order of $1\,\mu$s.

In order to apply the time-of-flight technique it is necessary to have some indication of the starting time. This is directly achieved if the spectrometer is used in connection with a pulsed accelerator as, for instance, a cyclotron. Another possible use is as a coincidence spectrometer, such that the starting time is indicated by γ-radiation emitted in the same decay as the β-particle. This method was used in a preliminary experiment which will be described in the paper following this one [3].

By any of these methods it is possible to measure a whole spectrum or a certain part of it simultaneously by keeping the magnetic field at a constant value and by studying the distribution of flight times by some multichannel method.

III. On the resolution

The main source of error which puts a limit to the resolution is associated with the time measurement. With fast scintillation detectors a flight time can be measured with a resolution, $\Delta\tau$, which is a few millimicroseconds. This corresponds to a resolution $\left(\frac{\Delta E}{E}\right)_1$ which is approximately equal to $\frac{\Delta\tau}{\tau}$, where τ is given by eq. (22).

$$\left(\frac{\Delta E}{E}\right)_1 = \frac{\Delta\tau}{\pi r_0} \cdot v \cdot \frac{n\varrho}{2 r_0}$$

With $n=2$, $r_0 = 50$ cm, $\Delta\tau = 2$ *ns* and $v = 3 \times 10^{10}$ cm/s

$$\left(\frac{\Delta E}{E}\right)_1 = \frac{8 \cdot \varrho}{1000}$$

If different starting directions from a point source are taken into consideration, the spread in flight path will be of the order of ϱ. This gives a contribution $\left(\frac{\Delta E}{E}\right)_2$ to the resolution which is $\approx \frac{\varrho}{\pi r_0}$

$$\left(\frac{\Delta E}{E}\right)_2 = \frac{6 \cdot \varrho}{1000}$$

This shows also that the source may have an extension in the aximuthal direction of the order of ϱ without appreciably affecting the resolution. As mentioned before, the extension in the radial direction has no influence on the resolution in the case where $n=2$.

IV. On the transmission

If a point source in the plane of symmetry of the magnet is considered which emits particles isotropically, the transmission of the instrument is easily calculable. All particles starting at a sufficiently small angle with the symmetry plane are accepted into the spectrometer. The maximum angle, ψ_0, is determined by the height of the pole-gap of the magnet.

Since the magnetic moment μ is a constant, we find

$$\mu=\frac{\mathcal{E}}{H_1}=\frac{\mathcal{E}}{H_0}\cdot\left(1-\frac{n^2}{2}\cdot\frac{z_1^2}{r_0^2}+\cdots\right) \quad \text{at the turning point and}$$

$$\mu=\frac{\mathcal{E}_\perp}{H_0}=\frac{\mathcal{E}}{H_0}\cdot\cos^2\psi \qquad \text{in the symmetry plane.}$$

This gives the following relation between ψ and z_1:

$$\sin\psi=\frac{n}{\sqrt{2}}\cdot\frac{z_1}{r_0}$$

If the height of the pole-gap is h, we find

$$\sin\psi_0=\frac{n}{2\sqrt{2}}\cdot\frac{h}{r_0},$$

which is the expression for the transmission. With $n=2$ and $r_0=50$ cm, a pole gap of 10 cm gives 14 % transmission.

V. On the application as a pair spectrometer

For several reasons it seems likely that the time-of-flight technique might be suitable for a pair spectrometer. The positive and negative electrons are directly separated since the drift motion goes in opposite directions in the two cases. By using a pulsed γ-source it is possible to measure the flight times of the two particles individually. In this way a wide range of energy distribution between the positive and the negative component of the pair can be utilized.

Due to the fact that the pair particles from a converter are strongly peaked in the direction of the γ-ray, both the resolution and the transmission are higher than in the case of an isotropic source. The resolution is improved because the spread in the flight path is reduced to a small fraction of the radius of curvature. If the γ-source and the converter are placed in the plane of symmetry of the magnet, the forward peaking of the pair particles increases the probability that the starting direction lies within the acceptance angle of the spectrometer. Since the forward peaking of the pair particles is the more pronounced the higher the γ-energy, practically all pairs formed in the converter are accepted into the spectrometer if the γ-energy is a few MeV.

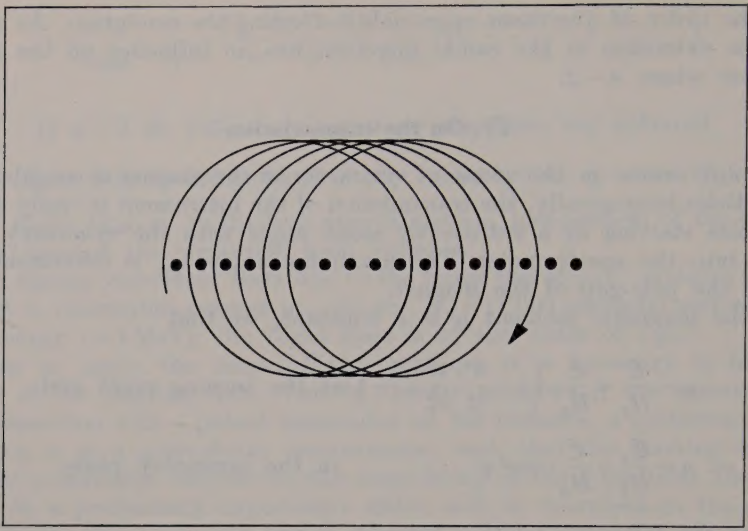

Fig. 1.

VI. On the possibility of selecting β-energies by means of a baffle system

The general idea of this spectrometer is to use the time-of-flight principle for measuring β-energies. It seems likely, however, that the instrument can be used also as an ordinary β-spectrometer by introducing a baffle which can be passed only by particles with a certain radius of curvature. In this way it seems possible to measure the energy of a pair with the time-of-flight technique without depending on a pulsed γ-source. The magnetic field and a baffle system define the energy of one of the pair particles and hence also its flight time from the source to the detector. By measuring the time difference between the signals from the two detectors it is now possible to calculate the flight time and the energy of the other particle. An analogous method may be used for measuring β-β-coincidences.

Such a filter can be based on the fact that the orbit is displaced a small distance, ΔL, during each revolution. According to eq. (5):

$$\Delta L = \frac{\pi \cdot n \cdot \varrho_0^2}{r_0}$$

If parallel equidistant metal wires are arranged perpendicular to the symmetry plane of the magnet at the mean radius r_0, β-particles can pass this gird only if the displacement, ΔL, is the same as the distance, a, between the wires. At the same time $2\varrho_0$ must be a whole number times a. Figure 1 shows the principle of such a baffle system. With $n=2$, $r_0=50$ cm and $\varrho_0 \approx 1$ cm one finds $a=0.125$ cm. The energy resolution can probably be rather high, depending on the length of the baffle. In the case of a pair spectrometer the transmission need not be severely reduced. Another type of baffle may possibly be based on the fact that the "wavelength" of the oscillation through the symmetry plane is proportional to the radius of curvature.

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